Cramer's Rule, Volume, and Linear Transformations

Cramer's Rule

Needed in a variety of theoretical calculations. For instance, to study how the solution of $A\vec{x} = \vec{b}$ is affected by changes in the entries of b.

However, the formula is innefficient for hand calculations, except for 2×2 or perhaps 3×3 matrices.

For any $A_{n \times n}$ and any $\vec{b} \in \mathbb{R}^n$, let $A_i(\vec{b})$ be the matrix obtained from A by replacing column i by the vector $\vec{b}$: $A_i(\vec{b}) = [a_1 \ldots \underset{\substack{\uparrow \\ col\, i}}{\vec{b}} \ldots a_n]$

Theorem. (Cramer's Rule)
Let $A_{n \times n}$ be invertible. For any $\vec{b} \in \mathbb{R}^n$, the unique solution $\vec{x}$ of $A\vec{x} = \vec{b}$ has entries given by

$$x_i = \frac{det\, A_i(b)}{det\, A}, \ i = 1, 2, \ldots, n$$

Proof. denote the columns of A by $\vec{a}_1, \ldots\ \vec{a}_n$ and the columns of the identity matrix $I_{n \times n}$ by $\vec{e}_1, \ldots,\ \vec{e}_n$. If $A\,\vec{x} = \vec{b}$, the definition of matrix multiplication shows that

$$AI_i(\,\vec{x}) = A[\,\vec{e}_1 \ldots\ \vec{x} \ldots\ \vec{e}_n] = [A\,\vec{e}_1 \ldots A\vec{x} \ldots A\,\vec{e}_n] =$$

$$= [\,\vec{a}_1 \ldots\ \vec{b} \ldots\ \vec{a}_n] = A_i(\,\vec{b})$$

By the multiplicative property of determinants,

$$det\,(AI_i(\,\vec{x})) = (det\,A)(det\,I_i(\vec{x})) = det\,A_i(b)$$

But $(det\,I_i(\vec{x})) = x_i$, hence $x_i = \frac{det\,A_i(b)}{det\,A}$

Application to Engeneering

A number of important engineering problems, particularly in electrical engineering and control theory, can be analyzed by *Laplace transforms*. This approach converts an appropriate system of linear differential equations into a system of linear algebraic equations whose coefficients involve a parameter.

Example. Consider the following system in which s is an unspecified parameter. Determine the values of s for which the system has a unique solution, and use Cramer's rule to describe the solution.

$$\begin{cases} 3sx_1 - 5x_2 = 3 \\ 9x_1 + 5sx_2 = 2 \end{cases}$$

Solution. View the system as $A\vec{x} = \vec{b}$. Then

$$A = \begin{bmatrix} 3s & 5 \\ 9 & 5s \end{bmatrix}, \ A_1(\vec{b}) = \begin{bmatrix} 3 & 5 \\ 2 & 5s \end{bmatrix}, \ A_2(\vec{b}) = \begin{bmatrix} 3s & 3 \\ 9 & 2 \end{bmatrix}$$

since

$$det A = 15s^2 - 45 = 15s(5-3)$$

the system has a unique solution precisely when $s \neq 0, 3$. For such an s, the solution is (x_1, x_2), where

$$x_1 = \frac{det\, A_1(\vec{b})}{det\, A} = \frac{15s-10}{15s(s-3)}; \quad x_2 = \frac{det\, A_2(\vec{b})}{det\, A} = \frac{3s+24}{3(s+2)(s-2)} = \frac{6s-27}{15s(s-3)}$$

Definition. The matrix of cofactors on the right side is called the ***adjugate*** (or ***classical adjoint***) of A, denoted by $adj\, A$

$$A^{-1} = \frac{1}{det\, A} \begin{bmatrix} C_{11} & C_{21} & . & . & . & C_{n1} \\ C_{12} & C_{22} & . & . & . & C_{n2} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ C_{1n} & C_{2n} & & & & C_{nn} \end{bmatrix}$$

<u>Theorem</u>. Let $A_{n \times n}$ be an invertible matrix. Then

$$\boxed{A^{-1} = \frac{1}{detA} adj\, A}$$

<u>Numerical Notes</u>

This theorem is useful mainly for theoretical calculations. The formula for A^{-1} permits one to deduce properties of the inverse without actually calculating it. Except for special cases, the algorithm gives a much better way to compute A^{-1}, if the inverse is really needed.

Cramer's rule is also a theoretical tool. It can be used to study how sensitive the solution of $A\vec{x} = \vec{b}$ is to changes in an entry in $\vec{b}$ or in A (perhaps due to experimental error when acquiring the entries for $\vec{b}$ or A). When A is a 3×3 matrix with complex entries, Cramer's rule is sometimes selected for hand-computation because row reduction of $[A|\vec{b}]$ with complex arithmetic can be messy, and the determinants are fairly easy to compute. For a larger $n \times n$ matrix (real or complex), Cramer's rule is hopelessly inefficient. <u>Computing just one determinant takes about as much work as solving $A\vec{x} = \vec{b}$ by row reduction</u>.

<u>Theorem.</u> If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|det\,A|$. If A is a 3×3 matrix, the volume of the prallelepiped determined by the columns of A is $|det\,A|$.

Let $\vec{a}_1$ and $\vec{a}_2$ be nonzero vectors. Then for any scalar c, the area of the parallelogram determined by $\vec{a}_1$ and $\vec{a}_2$ equals the area of the parallelogram determined by $\vec{a}_1$ and $\vec{a}_2 + c\vec{a}_1$.

Theorem. Let $T : \mathbb{R}^2 \to \mathbb{R}^2 : T(\vec{x}) = A_{2\times 2}(\vec{x})$ be a *l. t.* If S is a parallelogram in $\mathbb{R}^2$, then

$$\boxed{\{\text{area of } T(S)\} = |det\,A| \cdot \{\text{area of } S\}}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\boxed{\{\text{volume of } T(S)\} = |det\,A| \cdot \{\text{ volume of } S\}}$$

<u>Theorem</u>. (An Inverse Formula) Let $A_{n\times n}$ be an invertible matrix. Then

$$\boxed{A^{-1} = \frac{1}{detA} adj\, A}$$